Exercise 39

1. $$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

Method 1: $x = \frac{1}{t} \Rightarrow \frac{dx}{dt} = -t^{-2}$

$$= \int \frac{1}{t^{-2}\sqrt{1+t^{-2}}}\left(-t^{-2}\right) dt$$

$$= -\int \frac{1}{\sqrt{1+t^{-2}}} dt$$

$$= -\int \frac{1}{\sqrt{1+\frac{1}{t^2}}} dt \quad : \sqrt{1+\frac{1}{t^2}} = \sqrt{\frac{1+t^2}{t^2}} = \frac{\sqrt{1+t^2}}{t}$$

$$= -\int \frac{t}{\sqrt{1+t^2}} dt$$

Let $1+t^2 = u \Rightarrow \frac{du}{dt} = 2t$

$$= -\int \frac{1}{2} \frac{1}{\sqrt{u}} du$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

$$= -\frac{1}{2} \frac{1}{-\frac{1}{2}+1} u^{-\frac{1}{2}+1} + C$$

$$= -u^{\frac{1}{2}} + C$$

$$= -\sqrt{1+t^2} + C = -\sqrt{1+x^{-2}} + C = -\frac{\sqrt{1+x^2}}{x} + C$$

Method 2:

$x = \tan\theta, \quad \frac{dx}{d\theta} = \sec^2\theta$

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

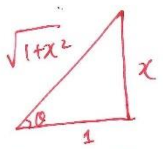
$$= \int \frac{1}{\tan^2\theta\sqrt{1+\tan^2\theta}} \sec^2\theta \, d\theta$$

$$= \int \frac{1}{\tan^2\theta \sec\theta} \sec^2\theta \, d\theta$$

$$= \int \frac{1}{\tan^2\theta} \sec\theta \, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta} \cdot \frac{1}{\cos\theta} \, d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta} \, d\theta$$



$\tan\theta = x$

$\Rightarrow \sin\theta = \frac{x}{\sqrt{1+x^2}}$

$u = \sin\theta, \quad \frac{du}{d\theta} = \cos\theta$

$$= \int \frac{1}{u^2} \, du$$

$$= -u^{-1} + C$$

$$= -(\sin\theta)^{-1} + C$$

$$= -\frac{\sqrt{1+x^2}}{x} + C$$

Method 3:

$$\int \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int \frac{1}{x^3\sqrt{1+\frac{1}{x^2}}}\,dx$$

Let $u = 1 + \frac{1}{x^2}$

$$\frac{du}{dx} = -2x^{-3}$$

then

$$= \int \frac{1}{\sqrt{u}\; x^3}\;\frac{du}{-2x^{-3}}$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$= -\frac{1}{2}\;\frac{1}{-\frac{1}{2}+1}\,(u)^{-\frac{1}{2}+1} + C$$

$$= -(u)^{\frac{1}{2}} + C$$

$$= -\sqrt{1+\frac{1}{x^2}} + C$$

$$= -\frac{\sqrt{1+x^2}}{x} + C$$